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Coherent states for a polynomial $su(1,1)$ algebra and a conditionally solvable system

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Abstract

In a previous paper (2007 *J. Phys. A: Math. Theor.* **40** 11105), we constructed a class of coherent states for a polynomially deformed $su(2)$ algebra. In this paper, we first prepare the discrete representations of the nonlinearly deformed $su(1, 1)$ algebra. Then we extend the previous procedure to construct a discrete class of coherent states for a polynomial $su(1, 1)$ algebra which contains the Barut–Girardello set and the Perelomov set of the $SU(1, 1)$ coherent states as special cases. We also construct coherent states for the cubic algebra related to the conditionally solvable radial oscillator problem.

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1. Introduction

In a previous paper [1], we have constructed a set of coherent states for a polynomially deformed $su(2)$ algebra. The goal of this paper is to construct a discrete class of coherent states for a polynomial $su(1, 1)$ algebra by extending the procedure employed for the polynomial $su(2)$ case. For the usual $SU(1, 1)$ group, there are two well-known sets of coherent states: the Barut–Girardello coherent states [2] which are characterized by the complex eigenvalues ξ of the non-compact generator $\hat{K}_-$ of the $su(1, 1)$ algebra

$$\hat{K}_-|\xi\rangle = \xi|\xi\rangle \tag{1}$$

and the Perelomov coherent states [3] which are characterized by points η of the coset space $SU(1, 1)/U(1)$

$$|\eta\rangle = N^{-1}\mathrm{e}^{\eta\hat{K}_+}|0\rangle, \qquad \hat{K}_-|0\rangle = 0. \tag{2}$$

These two sets are not equivalent. Since we have no knowledge of the group structure corresponding to the polynomial $su(1, 1)$ algebra, we are unable to follow Perelomov's group theoretical approach. Thus, we construct coherent states in such a way that they are reducible

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either to the Barut–Girardello $SU(1,1)$ states or the Perelomov $SU(1,1)$ states in the linear limit. In the literature [4–6], several authors have proposed various sets of coherent states for the polynomial $su(1,1)$ algebra in different contexts. What we wish to study here is a unified treatment of coherent states of the Barut–Girardello type and the Perelomov type for the polynomial $su(1,1)$, which differs from all of those reported earlier.

The polynomial $su(2)$ algebra we considered earlier [1] is a special case of the nonlinearly deformed $su(2)$ algebra of Bonatos, Danskaloyannis and Kolokotronis (BDK) [7]. BDK's deformed algebra, denoted by $su_\Phi(2)$, is of the form,

$$[\hat{J}_0, \hat{J}_\pm] = \pm\hat{J}_\pm, \qquad [\hat{J}_+, \hat{J}_-] = \Phi(\hat{J}_0(\hat{J}_0+1)) - \Phi(\hat{J}_0(\hat{J}_0-1)), \tag{3}$$

where the structure function $\Phi(x)$ is an increasing function of x defined for $x \geqslant -1/4$. The Casimir operator for $su_\Phi(2)$ is

$$\hat{\mathbf{J}}^2 = \hat{J}_-\hat{J}_+ + \Phi(\hat{J}_0(\hat{J}_0+1)) = \hat{J}_+\hat{J}_- + \Phi(\hat{J}_0(\hat{J}_0-1)). \tag{4}$$

On the basis $\{|j,m\rangle\}$ that diagonalizes $\hat{\mathbf{J}}^2$ and $\hat{J}_0$ simultaneously such that [7]

$$\hat{\mathbf{J}}^2|j,m\rangle = \Phi\left(j(j+1)\right)|j,m\rangle, \qquad \hat{J}_0|j,m\rangle = m|j,m\rangle, \tag{5}$$

the operators $\hat{J}_+$ and $\hat{J}_-$ satisfy the relations

$$\hat{J}_+|j,m\rangle = \sqrt{\Phi(j(j+1)) - \Phi(m(m+1))}|j,m+1\rangle \tag{6}$$

$$\hat{J}_-|j,m\rangle = \sqrt{\Phi(j(j+1)) - \Phi(m(m-1))}|j,m-1\rangle \tag{7}$$

with $2j = 0, 1, 2, \ldots,$ and $|m| \leqslant j$.

The coherent states we constructed for $su_\Phi(2)$ by letting $m = -j + n\,(n = 0, 1, 2, \ldots, 2j)$ were

$$|j,\xi\rangle = N_\Phi^{-1}(|\xi|)\sum_{n=0}^{2j}\frac{\sqrt{[k_n]!}}{n!}\xi^n|j,-j+n\rangle. \tag{8}$$

Here

$$k_n = \Phi(j(j+1)) - \Phi((j-n)(j-n+1)) \tag{9}$$

and

$$[k_n]! = \prod_{j=1}^{n}k_n, \qquad [k_0]! = 1. \tag{10}$$

The normalization factor was given by

$$N_\Phi^2(|\xi|) = \sum_{n=0}^{2j}\frac{[k_n]!|\xi|^{2n}}{(n!)^2}. \tag{11}$$

For our polynomial $su(2)$ case, we imposed the polynomial condition that

$$\Phi(x) = \sum_{r=1}^{p}\alpha_r x^r \qquad (\alpha_r \in \mathbf{R}) \tag{12}$$

with $\alpha_p \neq 0$. We showed that the coherent states we obtained include the usual $su(2)$ coherent states and the cubic $su(2)$ coherent states as special cases.

In this paper, we first extend BDK's $su_\Phi(2)$ to a nonlinearly deformed $su(1,1)$ algebra and prepare discrete representations for the algebra which correspond to those belonging to the positive discrete series of the irreducible unitary representations of $SU(1,1)$. Then we construct formally a set of coherent states for the deformed algebra $su_\Phi(1,1)$ by generalizing the $SU(1,1)$ group element used for the Perelomov states. As before, we also impose the polynomial condition (12) to specify the coherent states for the polynomially deformed algebra $su_{2p-1}(1,1)$. Out of the formal states so constructed, we select two sets of states which are reducible to the Barut–Girardello set and the Perelomov set in the linear limit. Finally, we reformulate the conditionally solvable radial oscillator problem in broken supersymmetric quantum mechanics, proposed by Junker and Roy [8], in an algebraic manner to show that the eigenstates of one of the partner Hamiltonians, $\hat{H}_+$, in SUSY quantum mechanics can be identified with a standard basis of the $su(1,1)$ algebra whereas the set of eigenstates of the other partner Hamiltonian $\hat{H}_-$ is identified with a representation space of the cubic algebra $su_3(1,1)$. We also construct coherent states of the Barut–Girardello type and of the Perelomov type for the conditionally solvable problem.

2. Polynomial $su(1,1)$ algebra and its representations

In order to introduce a nonlinearly deformed $su(1,1)$ algebra in a manner parallel to the nonlinearly deformed algebra $su_\Phi(2)$ of Bonatos, Danskaloyannis and Kolokotronis [7], we exercise analytic continuation [9–11] on $su_\Phi(2)$. Replacing the generators of $su_\Phi(2)$ in (3) as

$$\hat{J}_0 \to \hat{K}_0, \qquad \hat{J}_\pm \to \mathrm{i}\hat{K}_\pm, \tag{13}$$

we extend $su_\Phi(2)$ formally into a deformed $su(1,1)$ algebra,

$$[\hat{K}_0, \hat{K}_\pm] = \pm\hat{K}_\pm, \qquad [\hat{K}_+, \hat{K}_-] = \Phi(\hat{K}_0(\hat{K}_0 - 1)) - \Phi(\hat{K}_0(\hat{K}_0 + 1)), \tag{14}$$

which we denote by $su_\Phi(1,1)$ as an extension of BDK's $su_\Phi(2)$. Here, we assume that the generators of $su_\Phi(1,1)$ in (14) possess the Hermitian properties,

$$\hat{K}_0^\dagger = \hat{K}_0, \qquad \hat{K}_\pm^\dagger = \hat{K}_\mp. \tag{15}$$

We also assume that the structure function $\Phi(x)$ is a differentiable function increasing with a real variable $x \geqslant -1/4$, and is operator-valued and Hermitian when x is a Hermitian operator. Accordingly the operator obtainable from the Casimir operator (4) of $su_\Phi(2)$ by the analytic continuation (13),

$$\hat{\mathbf{K}}^2 = -\hat{K}_-\hat{K}_+ + \Phi(\hat{K}_0(\hat{K}_0 + 1)) = -\hat{K}_+\hat{K}_- + \Phi(\hat{K}_0(\hat{K}_0 - 1)), \tag{16}$$

is Hermitian. From the first equation of (14) immediately follows

$$\hat{K}_0^r\hat{K}_\pm = \hat{K}_\pm(\hat{K}_0 \pm 1)^r \tag{17}$$

for $r = 0, 1, 2, \ldots$. Since the structure function $\Phi(x)$, assumed to be a real differentiable function, can be expanded as a MacLaurin series, it is obvious that

$$\Phi(\hat{K}_0(\hat{K}_0 \mp 1))\hat{K}_\pm = \hat{K}_\pm\Phi(\hat{K}_0(\hat{K}_0 \pm 1)). \tag{18}$$

Therefore, the operator $\hat{\mathbf{K}}^2$ of (16), being commutable with all the three generators, is indeed the Casimir invariant of $su_\Phi(1,1)$.

By imposing the polynomial condition (12) on the structure function in (14), we obtain a polynomial $su(1,1)$ algebra,

$$[\hat{K}_0, \hat{K}_\pm] = \pm\hat{K}_\pm, \qquad [\hat{K}_+, \hat{K}_-] = -2\sum_{r=1}^{p}\alpha_r\hat{K}_0^r\sum_{s=1}^{r}(\hat{K}_0 + 1)^{r-s}(\hat{K}_0 - 1)^{s-1}. \tag{19}$$

When $\Phi(x)$ is a polynomial in $x = \hat{K}_0(\hat{K}_0 + 1)$ of degree p, the right-hand side of the second equation of (19) becomes a polynomial in $\hat{K}_0$ of degree $2p - 1$. Thus (19) is the polynomial $su(1,1)$ algebra of odd degree $2p - 1$ ($p = 1, 2, 3, \ldots$), which we denote by

$su_{2p-1}(1, 1)$. As special cases, $p = 1$ and $p = 2$ correspond to the usual $su(1, 1)$ and the cubic algebra $su_{cub}(1, 1)$, respectively. The present scheme cannot accommodate polynomial $su(1, 1)$ algebras of even degree.

In analogy with the case of $su_\Phi(2)$ represented on the basis $\{|j, m\rangle\}$ as in (5), we consider a representation space for $su_\Phi(1, 1)$ which is spanned by simultaneous eigenstates $\{|k, m\rangle\}$ of the Casimir operator $\hat{\mathbf{K}}^2$ and the compact operator $\hat{K}_0$. On the basis $\{|k, m\rangle\}$, let $\hat{\mathbf{K}}^2$ and $\hat{K}_0$ be diagonalized as

$$\hat{\mathbf{K}}^2|k, m\rangle = \Phi(k(k-1))|k, m\rangle, \qquad \hat{K}_0|k, m\rangle = m|k, m\rangle. \tag{20}$$

From the relations (15), (16) and (17), it is clear that the operators $\hat{K}_\pm$ act on the above states as

$$\hat{K}_+|k, m\rangle = \sqrt{\Phi(m(m+1)) - \Phi(k(k-1))}|k, m+1\rangle, \tag{21}$$

$$\hat{K}_-|k, m\rangle = \sqrt{\Phi(m(m-1)) - \Phi(k(k-1))}|k, m-1\rangle. \tag{22}$$

For the usual $su(1, 1)$ case ($p = 1$), we wish to take the basis states $|k, m\rangle$ from those of the unitary irreducible representations of the group $SU(1, 1)$. As is well known, the representations of $SU(1, 1)$ are classified into [11–13]: (i) the positive discrete series $D_n^+(k)$, (ii) the negative discrete series $D_n^-(k)$, (iii) the principle continuous series $C_n(m_0, k)$ and (iv) the supplementary continuous series $E_n(m_0, k)$. As for the polynomial $su(1, 1)$, however, the corresponding group and its representations are not available. In the present work, we are only interested in constructing a set of coherent states for discrete dynamics of the polynomial $su(1, 1)$. Therefore, we examine whether the positive discrete series $D_n^+(k)$ of $SU(1, 1)$, for which

$$k \in \mathbf{R}^+, \qquad m - k \in \mathbf{N}_0, \tag{23}$$

is compatible with $su_\Phi(1, 1)$. Here, we have used the notation $\mathbf{N}_0 = \mathbf{N} \cup \{0\} = \{0, 1, 2, 3, \ldots\}$.

The compact generator $\hat{K}_0$ has been chosen to be Hermitian so that its eigenvalues m are real. As the property (17) for $r = 1$ indicates, the operators $\hat{K}_\pm$ map eigenstates $|k, m\rangle$ of $\hat{K}_0$ into $|k, m \pm 1\rangle$, respectively. Hence the value of m increases or decreases by integer units as

$$m = m_0 + n, \tag{24}$$

where $m_0 \in \mathbf{R}$ and $n \in \mathbf{Z}$. The eigenvalue $\Phi(k(k-1))$ of the Casimir operator $\hat{\mathbf{K}}^2$ must be real. In fact, the structure function has been assumed to be a real function increasing with its argument greater than or equal to $-1/4$. Therefore, k must satisfy the conditions

$$k(k-1) \in \mathbf{R} \qquad \text{and} \qquad \left(k - \tfrac{1}{2}\right)^2 \geqslant 0, \tag{25}$$

from which follows

$$k \in \mathbf{R}. \tag{26}$$

Furthermore, (15) yields

$$\langle k, m|\hat{K}_\pm^\dagger \hat{K}_\pm|k, m\rangle = \langle k, m|\hat{K}_\mp \hat{K}_\pm|k, m\rangle \geqslant 0, \tag{27}$$

and (16) and (20) lead to

$$\Phi\,(m(m \pm 1)) - \Phi\,(k(k-1)) \geqslant 0. \tag{28}$$

As $\Phi(x)$ is an increasing function, the $SU(1, 1)$ discrete series (23) satisfies these conditions with $m_0 = k$. Thus, we may choose as the basis $\{|k, m\rangle\}$ for $su_\Phi(1, 1)$

$$k \in \mathbf{R}^+, \qquad m = k + n(n \in \mathbf{N}_0). \tag{29}$$

In the above analysis, we have not explicitly used the polynomial condition (12) even though the structure function $\Phi(x)$ was assumed to be expressible as a MacLaurin series of x.

In view of the basis chosen above, we realize that it is more convenient to characterize the basis states by means of the integral number n rather than $m = k + n$. Thus, we let the orthonormalized set $\{|k, n\rangle\}$ span the representation space with $k \in \mathbf{R}^+$ and $n \in \mathbf{N}_0$. On this basis, we rewrite (20), (21) and (22) as

$$\hat{K}_0|k, n\rangle = (k + n)|k, n\rangle, \tag{30}$$

$$\hat{K}_+|k, n\rangle = \sqrt{\phi_{n+1}(k)}|k, n + 1\rangle, \tag{31}$$

$$\hat{K}_-|k, n\rangle = \sqrt{\phi_n(k)}|k, n - 1\rangle, \tag{32}$$

where we have introduced the shorthand notation,

$$\phi_n(k) = \Phi((k + n)(k + n - 1)) - \Phi(k(k - 1)), \tag{33}$$

which we shall call the structure factor for convenience. From (32) it is evident that

$$\hat{K}_-|k, 0\rangle = 0. \tag{34}$$

Hence, $|k, 0\rangle$ can be taken as the fiducial state. Also from (31) follows that

$$|k, n\rangle = \frac{1}{\sqrt{[\phi_n(k)]!}}(\hat{K}_+)^n|k, 0\rangle. \tag{35}$$

In the above, we have used the generalized factorial notation signifying

$$[\phi_n(k)]! = \prod_{l=1}^{n} \phi_l(k), \qquad [\phi_0(k)]! = 1, \tag{36}$$

which will also be used later for other sequences of functions. Furthermore, for simplicity, we express $\phi_n(k)$ by ϕ_n.

3. Coherent states for $su_\Phi(1, 1)$

Now we wish to construct generalized coherent states for $su_\Phi(1, 1)$ which accommodate those of the Barut–Girardello type and the Perelomov type as special cases. By the Barut–Girardello type (BG-type) and the Perelomov type (P-type), we mean the coherent states for the nonlinear $su(1, 1)$ which are reducible to the Barut–Girardello $SU(1, 1)$ coherent states and the Perelomov $SU(1, 1)$ coherent states in the linear limit, respectively.

3.1. Generalized coherent states

First, we introduce a generalized exponential function,

$$[\mathrm{e}(\nu)]^x = \sum_{n=0}^{\infty} \frac{x^n}{[\nu_n]!} \tag{37}$$

defined on a base sequence $\{\nu_1, \nu_2, \ldots, \nu_n\}$ with $\lim_{n\to\infty} |\nu_n| \neq 0$. Then we consider a set of states constructed on the fiducial state (34) as

$$|k, \zeta\rangle = N_\Phi^{-1}(|\zeta|)[\mathrm{e}(\nu)]^{\zeta \hat{K}_+}|k, 0\rangle, \tag{38}$$

where $\zeta \in \mathbf{C}$. This is similar in the form to the definition of the Perelomov $SU(1, 1)$ coherent states (2). However, we take this as a unified treatment of the BG-type and the P-type. By the definition of the generalized exponential function (37), the state (38) is expressed as

$$|k, \zeta\rangle = N_\Phi^{-1}(|\zeta|) \sum_{n=0}^{\infty} \frac{(\zeta \hat{K}_+)^n}{[\nu_n]!}|k, 0\rangle. \tag{39}$$

Use of (35) further leads (39) to an alternative form

$$|k, \zeta\rangle = N_{\Phi}^{-1}(|\zeta|) \sum_{n=0}^{\infty} \frac{\sqrt{[\phi_n]!}}{[\nu_n]!} \zeta^n |k, n\rangle. \tag{40}$$

These states are normalized to unity with

$$|N_{\Phi}(|\zeta|)|^2 = \sum_{n=0}^{\infty} \frac{[\phi_n]!}{([\nu_n]!)^2} |\zeta|^{2n}. \tag{41}$$

Here, the radius of convergence is

$$R = \lim_{n\to\infty} \frac{|\nu_n|^2}{|\phi_n|}. \tag{42}$$

The states (40), parameterized by a continuous complex number ζ, share a number of the properties that the coherent states are to possess. They are not in general orthogonal. From the Schwarz inequality, we have

$$\langle k, \zeta | k, \zeta'\rangle = N_{\Phi}^{*-1}(|\zeta|) N_{\Phi}^{-1}(|\zeta'|) \sum_{n=0}^{\infty} \frac{[\phi_n]!}{([\nu_n]!)^2} (\zeta^* \zeta')^n \leqslant 1, \tag{43}$$

which is not zero when $\zeta \neq \zeta'$. They resolve unity,

$$\hat{1} = \int d\mu(\zeta, \zeta^*) |k, \zeta\rangle\langle k, \zeta|, \tag{44}$$

if the integration measure can be found in the form

$$d\mu(\zeta, \zeta^*) = \frac{1}{2\pi} |N_{\Phi}(|\zeta|)|^2 \rho(|\zeta|^2)\, d|\zeta|^2\, d\varphi. \tag{45}$$

Here $\zeta = |\zeta| e^{i\varphi} (0 \leqslant \varphi < 2\pi)$, and the weight function $\rho(|\zeta|^2)$ is to be determined by its moments,

$$\int_0^{\infty} \rho(t) t^n\, dt = \frac{([\nu_n]!)^2}{[\phi_n]!}, \tag{46}$$

where we have let $t = |\zeta|^2$. The non-orthogonality (43) together with the resolution of unity (44) shows that the states form an overcomplete basis in the representation space spanned by the discrete eigenstates of the compact operator $\hat{K}_0$ bounded below. Note also that these states are temporally stable for a system with the Hamiltonian $\hat{H} = \hbar\omega(\hat{K}_0 - k)$ as the states (40) evolve according to

$$e^{-i\hat{H}t/\hbar} |k, \zeta\rangle = |k, \zeta e^{-i\omega t}\rangle. \tag{47}$$

With these properties, the states constructed in (40) may be considered as generalized coherent states for $su_{\Phi}(1, 1)$.

3.2. Coherent states for $su_{2p-1}(1,1)$

Next, we impose on $su_{\Phi}(1, 1)$ the polynomial condition,

$$\Phi(x) = \sum_{r=1}^{p} \alpha_r x^r \qquad (\alpha_r \in \mathbf{R}), \tag{48}$$

where $\alpha_1 > 0, \alpha_p \neq 0, d\Phi/dx > 0$ and $x \geqslant -1/4$. This is the same as (12) applied to $su_{\Phi}(2)$. Under this condition, $su_{\Phi}(1, 1)$ becomes a polynomial $su(1, 1)$ algebra of order $2p-1$, which we denote by $su_{2p-1}(1, 1)$. In the limit that $\alpha_r \to 0$ for $r = 2, 3, \ldots, p$, the structure function for $p = 1$ becomes $\Phi(x) = \alpha_1 x$. In the resultant linear algebra $su_1(1,1)$, we can let $\alpha_1 = 1$ without loss of generality. Thus, we identify $su_1(1,1)$ with the usual linear $su(1,1)$ algebra.

For $su_{2p-1}(1,1)$, the structure factor ϕ_n of (33) takes the form

$$\phi_n = \sum_{r=1}^{p} \alpha_r[(k+n)^r(k+n-1)^r - k^r(k-1)^r] = n(2k+n-1)\chi_n, \tag{49}$$

where

$$\chi_n = \sum_{r=1}^{p}\sum_{s=1}^{r} \alpha_r[k(k-1)]^{r-s}[(k+n)(k+n-1)]^{s-1}. \tag{50}$$

Note that for large n

$$\chi_n \sim O(n^{2p-2}), \qquad \phi_n \sim O(n^{2p}). \tag{51}$$

It is evident that $\chi_n = \alpha_1$ and $\phi_n = n(2k+n-1)$ for $p = 1$. This means that χ_n for $p > 1$ characterizes the nonlinear deformation of $su_{2p-1}(1,1)$. In this regard, we refer to χ_n as the deformation factor.

The generalized factorial of ϕ_n given by (49) is

$$[\phi_n]! = n!(2k)_n[\chi_n]!, \tag{52}$$

where used is the Pochhammer symbol

$$(z)_n = \frac{\Gamma(z+n)}{\Gamma(z)} = (-1)^n \frac{\Gamma(1-z)}{\Gamma(1-z-n)}. \tag{53}$$

The deformation factor χ_n of (50) is an inhomogeneous polynomial of degree $2p-2$ which can be written as

$$\chi_n = \alpha_p \prod_{i=1}^{2p-2}(n-a_i), \tag{54}$$

where a_i's are the roots of $\chi_n = 0$ with respect to n. Its generalized factorial can be expressed as

$$[\chi_n]! = \chi_1\chi_2\ldots\chi_n = \alpha_p^n \prod_{i=1}^{2p-2}(1-a_i)_n. \tag{55}$$

Substitution of (55) into (52) yields

$$[\phi_n]! = \alpha_p^n n!(2k)_n \prod_{i=1}^{2p-2}(1-a_i)_n. \tag{56}$$

Inserting (52) into (40) and (41), we obtain a formal expression for the coherent states for the polynomial algebra $su_{2p-1}(1,1)$,

$$|k,\zeta\rangle = N_p^{-1}(|\xi|)\sum_{n=0}^{\infty} \frac{\sqrt{n!(2k)_n[\chi_n]!}}{[\nu_n]!}\zeta^n|k,n\rangle \tag{57}$$

and

$$|N_p(|\zeta|)|^2 = \sum_{n=0}^{\infty} \frac{n!(2k)_n[\chi_n]!}{([\nu_n]!)^2}|\zeta|^{2n}. \tag{58}$$

The coherent states (57) remain to be formal until $[\nu_n]!$ is specified. In order to accommodate the set of $SU(1,1)$ coherent states as a limiting case, we have to choose appropriately $[\nu_n]!$. In the proceeding sections, we specifically consider two cases: the Barut–Girardello type (BG-type) whose states go over to the Barut–Girardello $SU(1,1)$ states in the linear limit $(p = 1)$ and the Perelomov type (P-type) whose coherent states approach the Perelomov $SU(1,1)$ states in the same limit.

4. Coherent states of the Barut–Girardello type

Out of the generalized coherent states (57) formally constructed for the $su_{2p-1}(1,1)$, we select the BG-type states by letting

$$\nu_n = \phi_n. \tag{59}$$

With this choice, (40) reads

$$|k,\xi\rangle = N_p^{-1}(|\xi|)\sum_{n=0}^{\infty}\frac{\xi^n}{\sqrt{[\phi_n]!}}|k,n\rangle, \tag{60}$$

where we have let $\zeta = \xi$ for the BG-type. Because of (51), the radius of convergence of (60) is infinity. This means that the BG-type states (60) can be defined on the full complex plane of ξ. It is easy to verify by utilizing (31) that the coherent states (60) are indeed eigenstates of the non-Hermitian operator $\hat{K}_+$,

$$\hat{K}_+|k,\xi\rangle = \xi|k,\xi\rangle, \tag{61}$$

with complex eigenvalues ξ. More explicitly, substitution of (56) into (60) yields

$$|k,\xi\rangle = N_p^{-1}(|\xi|)\sum_{n=0}^{\infty}\left\{\alpha_p^n n!(2k)_n\prod_{i=1}^{2p-2}(1-a_i)_n\right\}^{-1/2}\xi^n|k,n\rangle. \tag{62}$$

The normalization factor is

$$|N_p(|\xi|)|^2 = \sum_{n=0}^{\infty}\frac{1}{n!(2k)_n\prod_{i=1}^{2p-2}(1-a_i)_n}\left(\frac{|\xi|^2}{\alpha_p}\right)^n \tag{63}$$

which can be expressed in the closed form as

$$|N_p(|\xi|)|^2 = {}_0F_{2p-1}(2k, 1-a_1, 1-a_2, \ldots, 1-a_{2p-2}; |\xi|^2/\alpha_p), \tag{64}$$

where ${}_pF_q$ is Pochhammer's generalized hypergeometric function defined by

$${}_pF_q(\alpha_1,\alpha_2,\ldots,\alpha_p;\gamma_1,\gamma_2,\ldots,\gamma_q;z) = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n(\alpha_2)_n\cdots(\alpha_p)_n}{(\gamma_1)_n(\gamma_2)_n\cdots(\gamma_q)_n}\frac{z^n}{n!}.$$

The hypergeometric series ${}_0F_q$ is analytic at any z. Hence, the normalization factor (64) is convergent for all values of $|\xi|^2/\alpha_p$.

The inner product of two such states takes the form

$$\langle k,\xi|k,\xi'\rangle = N_p^{*-1}(|\xi|)N_p^{-1}(|\xi'|){}_0F_{2p-1}(2k, 1-a_1, 1-a_2, \ldots, 1-a_{2p-2}; \xi^*\xi'/\alpha_p). \tag{65}$$

The coherent states thus constructed for $su_{2p-1}(1,1)$ are able to resolve unity if the weight function $\rho(|\xi|^2)$ is determined as follows. Inserting (56) into (46) we obtain

$$\int_0^{\infty}\rho(t)t^n\,\mathrm{d}t = \alpha_p^n n!(2k)_n\prod_{i=1}^{2p-2}(1-a_i)_n \tag{66}$$

or rewriting with $n = s-1$

$$\int_0^{\infty}\rho(t)t^{s-1}\,\mathrm{d}t = (\alpha_p)^{s-1}\frac{\Gamma(s)\Gamma(2k-1+s)\Gamma(-a_1+s)\Gamma(-a_2+s)\cdots\Gamma(-a_{2p-2}+s)}{\Gamma(2k)\Gamma(1-a_1)\Gamma(1-a_2)\cdots\Gamma(1-a_{2p-2})}, \tag{67}$$

from which the weight function can be found by the inverse Mellin transformation (see formula 7.811.4 in [18]) in terms of Meijer's G-function as

$$\rho(|\xi|^2) = \left[\alpha_p\Gamma(2k)\prod_{i=1}^{2p-2}\Gamma(1-a_i)\right]^{-1}G_{0\,2p}^{2p\,0}\left(\frac{|\xi|^2}{\alpha_p}\middle|0, 2k-1, -a_1, -a_2, \ldots, -a_{2p-2}\right). \tag{68}$$

With the weight function (68) for the measure (45) the resolution of unity (44) can be achieved.

So far we have selected the BG-type coherent states (62) out of the generalized coherent states (40). It is rather straightforward to show that the constructed states (62) are indeed reducible to the Barut–Girardello $SU(1,1)$ states in the linear limit. If the deformation factor tends to unity, i.e., $\chi_n \to 1$, then $[\phi_n]! \to n!(2k)_n$. For $p = 1$ and $\alpha_1 = 1$, the normalization factor (64) takes the form,

$$|N_1(|\xi|^2)|^2 = {}_0F_1(2k; |\xi|^2) = \Gamma(2k)|\xi|^{1-2k} I_{2k-1}(2|\xi|), \tag{69}$$

where $I_\nu(z)$ is the modified Bessel function of the first kind. Thus, in the linear limit the coherent states (62) become

$$|k,\xi\rangle = N_1^{-1}(|\xi|)\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!(2k)_n}}\xi^n|k,n\rangle. \tag{70}$$

The coherent states (70) are indeed the Barut–Girardello $SU(1,1)$ coherent states [2]. The weight function that enables the states (70) to resolve the unity follows from

$$\int_0^\infty \rho(t)t^{s-1}\,\mathrm{d}t = \Gamma(s)(2k)_{s-1}, \tag{71}$$

the result being

$$\rho(|\xi|^2) = \frac{1}{\Gamma(2k)}G_{02}^{20}(|\xi|^2|0, 2k-1) = \frac{2|\xi|^{2k-1}}{\Gamma(2k)}K_{2k-1}(2|\xi|), \tag{72}$$

where $K_\nu(z)$ is the modified Bessel function of the second kind.

5. Coherent states of the Perelomov type

Our next task is to construct a set of the Perelomov-type states from (57). To this end, we choose

$$\nu_n = n\chi_n \tag{73}$$

and let $\zeta = \eta$ to write (57) in the form,

$$|k,\eta\rangle = N_p^{-1}(|\eta|)\sum_{n=0}^{\infty}\frac{\sqrt{[\phi_n]!}}{n![\chi_n]!}\eta^n|k,n\rangle, \tag{74}$$

or, using (52),

$$|k,\eta\rangle = N_p^{-1}(|\eta|)\sum_{n=0}^{\infty}\sqrt{\frac{(2k)_n}{n![\chi_n]!}}\eta^n|k,n\rangle. \tag{75}$$

The radius of convergence for (75) is obtained by

$$R = \lim_{n\to\infty}\frac{|n\chi_n|^2}{n(2k+n)|\chi_n|} = \lim_{n\to\infty}|\chi_n|, \tag{76}$$

whose result depends on the parameter p. Since $\chi_n = \alpha_1$ for $p = 1$, the radius of convergence is finite, i.e., $R = \alpha_1$. If $p \neq 1$, again from (51), the radius R becomes infinity. Substitution of (56) and (54) converts (75) into

$$|k,\eta\rangle = N_p^{-1}(|\eta|)\sum_{n=0}^{\infty}\left[\frac{(2k)_n}{n!\prod_{j=1}^{2p-2}(1-a_j)_n}\right]^{1/2}\left(\frac{\eta}{\sqrt{\alpha_p}}\right)^n|k,n\rangle, \tag{77}$$

where the normalization factor in (77) is given by

$$|N_p(|\eta|)|^2 = {}_1F_{2p-2}(2k; 1-a_1, 1-a_2, \ldots, 1-a_{2p-2}; |\eta|^2/\alpha_p) \tag{78}$$

which is convergent for any real value of $|\eta|^2/\alpha_p$ if $p > 1$. The weight function $\rho(|\eta|^2)$ needed to resolve the unity can be determined by

$$\int_0^\infty \rho(t)t^n \, \mathrm{d}t = \frac{n![\chi_n]!}{(2k)_n}. \tag{79}$$

Utilizing $[\chi_n]!$ of (55) and letting $n = s-1$, we rewrite this as

$$\int_0^\infty \rho(t)t^{s-1} \, \mathrm{d}t = \frac{\Gamma(2k)}{\prod_{j=1}^{2p-2} \Gamma(1-a_j)} \frac{\alpha_p^{s-1}\Gamma(s)}{\Gamma(2k-1+s)} \prod_{j=1}^{2p-2} \Gamma(-a_j+s) \tag{80}$$

from which we obtain the weight function

$$\rho(|\eta|^2) = \frac{\Gamma(2k)}{\alpha_p \prod_{j=1}^{2p-2} \Gamma(1-a_j)} G^{2p-1\,0}_{1\,2p-1}\left(\frac{|\eta|^2}{\alpha_p} \middle| \begin{matrix} 2k-1 \\ 0, -a_1, -a_2, \ldots, -a_{2p-2} \end{matrix}\right) \tag{81}$$

valid for all values of $|\eta|^2/\alpha_p$ if $p > 1$, and for $0 < |\eta|^2/\alpha_1 < 1$ if $p = 1$.

In the linear limit $\chi_n \to \alpha_1$, the normalization factor (78) tends to

$$|N_1(|\eta|)|^2 = {}_1F_0(2k; |\eta|^2/\alpha_1) = (1 - |\eta|^2/\alpha_1)^{-2k}. \tag{82}$$

Therefore, the coherent states (77) become

$$|k, \eta\rangle = (1 - |\eta|^2/\alpha_1)^k \sum_{n=0}^{\infty} \left[\frac{(2k)_n}{n!}\right]^{1/2} \left(\frac{\eta}{\sqrt{\alpha_1}}\right)^n |k, n\rangle. \tag{83}$$

With $\alpha_1 = 1$, the last expression (83) coincides with Perelomov's result for the $SU(1, 1)$ coherent states [3]. For $p = 1$ and $\alpha_1 = 1$, the weight function (81) reduces to

$$\rho(|\eta|^2) = \Gamma(2k) G^{10}_{11}\left(|\eta|^2 \middle| \begin{matrix} 2k-1 \\ 0 \end{matrix}\right). \tag{84}$$

With the help of the identity

$$G^{10}_{11}\left(z \middle| \begin{matrix} 2k-1 \\ 0 \end{matrix}\right) = \frac{1}{\Gamma(2k-1)} {}_1F_0(2-2k; z) = \frac{1}{\Gamma(2k-1)} (1-z)^{2k-2}, \tag{85}$$

valid for $0 < |z| < 1$, the weight function can be simplified to the form

$$\rho(|\eta|^2) = (2k-1)\left(1 - |\eta|^2\right)^{2k-2} \tag{86}$$

which is defined only on the Poincaré disk. Furthermore, in order for the weight function to remain positive, it is necessary to demand that $2k > 1$.

6. Coherent states for the cubic algebra

In this section, we study the cubic case in more detail with one of the conditionally solvable problems in supersymmetric (SUSY) quantum mechanics proposed by Junker and Roy [8].

6.1. The cubic su(1, 1) algebra

The cubic $su(1, 1)$ algebra ($p = 2$) is the simplest special case of the odd-polynomial $su_{2p-1}(1, 1)$ for which the structure function is quadratic,

$$\Phi(x) = \alpha_1 x + \alpha_2 x^2, \tag{87}$$

where $\alpha_1 > 0$ and $\alpha_2 \neq 0$. The deformed algebra (14) with this quadratic structure function becomes a cubic algebra of the form

$$[\hat{K}_0, \hat{K}_\pm] = \pm\hat{K}_\pm, \qquad [\hat{K}_+, \hat{K}_-] = -2\alpha_1\hat{K}_0 - 4\alpha_2\hat{K}_0^3. \tag{88}$$

The deformation factor for the cubic algebra is

$$\chi_n = \alpha_1 + \alpha_2\{(k+n)(k+n-1) + k(k-1)\}, \tag{89}$$

which can be written as

$$\chi_n = \alpha_2(n - a_+)(n - a_-) \tag{90}$$

with the roots

$$a_\pm = -\frac{1}{2}(2k-1) \pm \frac{1}{2}\left\{2 - (2k-1)^2 - 4\frac{\alpha_1}{\alpha_2}\right\}^{1/2}. \tag{91}$$

Hence, the structure factor defined by (49) reads

$$\phi_n(k) = \alpha_2 n(2k+n-1)(n-a_+)(n-a_-), \tag{92}$$

with which the ladder operators $\hat{K}_+$ and $\hat{K}_-$ work in the representation space of $su_3(1, 1)$ as

$$\hat{K}_+|k, n\rangle = \sqrt{\alpha_2(n+1)(2k+n)(n+1-a_+)(n+1-a_-)}|k, n+1\rangle \tag{93}$$

$$\hat{K}_-|k, n\rangle = \sqrt{\alpha_2 n(2k+n-1)(n-a_+)(n-a_-)}|k, n-1\rangle. \tag{94}$$

The coherent states of the BG-type and the P-type can be constructed straightforwardly for the cubic algebra.

6.2. Conditionally solvable problems

At this point, we reformulate the conditionally solvable broken SUSY problem in [8] in a way appropriate to the present polynomial $su(1, 1)$ scheme.

In SUSY quantum mechanics (see, e.g., [14]), the partner Hamiltonians are given by

$$\hat{H}_\pm = \tfrac{1}{2}\hat{p}^2 + V_\pm(\hat{x}). \tag{95}$$

The partner potentials are expressed in terms of the SUSY potential $W(x)$ as

$$V_\pm(x) = \tfrac{1}{2}\{W^2(\hat{x}) \pm \mathrm{i}[\hat{p}, W(\hat{x})]\}, \tag{96}$$

where $[\hat{x}, \hat{p}] = \mathrm{i}(\hbar = 1)$. The partner Hamiltonians (95) may also be written as

$$\hat{H}_+ = \hat{A}\hat{A}^\dagger, \qquad H_- = \hat{A}^\dagger\hat{A}, \tag{97}$$

where

$$\hat{A} = \frac{1}{\sqrt{2}}(\mathrm{i}\hat{p} + W(\hat{x})), \qquad \hat{A}^\dagger = \frac{1}{\sqrt{2}}(-\mathrm{i}\hat{p} + W(\hat{x})). \tag{98}$$

Let the partner eigenequations be expressed by

$$\hat{H}_\pm\big|\psi_n^{(\pm)}\big\rangle = E_n^{(\pm)}\big|\psi_n^{(\pm)}\big\rangle, \qquad n = 0, 1, 2, \ldots. \tag{99}$$

If SUSY is broken [14],

$$E_n^{(+)} = E_n^{(-)} > 0 \tag{100}$$

and

$$\hat{A}^\dagger|\psi_n^{(+)}\rangle = \sqrt{E_n^{(+)}}|\psi_n^{(-)}\rangle, \qquad \hat{A}|\psi_n^{(-)}\rangle = \sqrt{E_n^{(-)}}|\psi_n^{(+)}\rangle. \tag{101}$$

By definition, for conditionally solvable problems [8], the SUSY potential $W(\hat{x})$ is separable to two parts as

$$W(\hat{x}) = U(\hat{x}) + f(\hat{x}), \tag{102}$$

where $U(x)$ is a shape-invariant SUSY potential and $f(x)$ is a function satisfying the equation,

$$f^2(\hat{x}) + 2U(\hat{x})f(\hat{x}) + \mathrm{i}[\hat{p}, f(\hat{x})] = 2(\varepsilon - 1), \tag{103}$$

ε being the adjustable parameter a certain value of which makes the problem solvable. The partner potentials are written as

$$V_+(\hat{x}) = \tfrac{1}{2}(U^2(\hat{x}) + \mathrm{i}[\hat{p}, U(\hat{x})]) + \varepsilon - 1, \tag{104}$$

$$V_-(\hat{x}) = \tfrac{1}{2}(U^2(\hat{x}) - \mathrm{i}[\hat{p}, U(\hat{x})]) - \mathrm{i}[\hat{p}, f(\hat{x})] + \varepsilon - 1. \tag{105}$$

Since $V_+(\hat{x})$ is a shape-invariant potential, the system of $\hat{H}_+$ is exactly solvable. The potential $V_-(\hat{x})$ is not shape invariant, but the eigenvalue problem with $\hat{H}_-$ becomes conditionally solvable.

As a specific example, we take, as in [8], a modified radial harmonic oscillator with broken SUSY, for which

$$U(x) = x + \frac{\gamma + 1}{x} \qquad (\gamma \geqslant 0) \tag{106}$$

and

$$f(x) = \frac{\mathrm{d}}{\mathrm{d}x} \ln {}_1F_1\left(\frac{1}{2} - \frac{\varepsilon}{2}, \gamma + \frac{3}{2}; -x^2\right) \tag{107}$$

in the coordinate representation. In order for the confluent hypergeometric function to be convergent for the whole range of x, the parameter ε must be subjected to the condition

$$\varepsilon + 2\varepsilon\gamma + 2 > 0. \tag{108}$$

This is indeed the condition on ε under which the modified oscillator becomes exactly solvable.

The potential $V_+(x)$ composed of the SUSY potential (106) is

$$V_+(x) = \frac{1}{2}x^2 + \frac{\gamma(\gamma + 1)}{x^2} + \gamma + \varepsilon + \frac{1}{2}, \tag{109}$$

which is shape invariant by choice. Although the exact energy spectrum of the Hamiltonian $\hat{H}_+$ can be calculated by the standard Gendenstein procedure [15] or by using the semiclassical broken SUSY formula [16], we employ here an algebraic approach [11, 13]. To this end, we introduce the following operators,

$$\begin{aligned} \hat{C}_0 &= \frac{1}{2}\left(\hat{H}_+ - g\hat{1}\right) \\ \hat{C}_1 &= \frac{1}{4}\left(\hat{p}^2 - \hat{x}^2 + \frac{\gamma(\gamma + 1)}{\hat{x}^2}\right) \\ \hat{C}_2 &= \frac{1}{4}(\hat{x}\hat{p} + \hat{p}\hat{x}) \end{aligned} \tag{110}$$

where $g = \gamma + \varepsilon + 1/2$. It is then easy to show that they obey the $su(1, 1)$ algebra,

$$[\hat{C}_0, \hat{C}_\pm] = \pm\hat{C}_\pm, \qquad [\hat{C}_+, \hat{C}_-] = -2\hat{C}_0. \tag{111}$$

where $\hat{C}_\pm \equiv \hat{C}_1 \pm \mathrm{i}\hat{C}_2$. The Casimir operator is

$$\hat{\mathbf{C}}^2 \equiv \hat{C}_0^2 - \hat{C}_1^2 - \hat{C}_2^2 \tag{112}$$

which turns out to be

$$\hat{\mathbf{C}}^2 = \frac{4\gamma(\gamma+1)-3}{16}\hat{1}. \tag{113}$$

On the basis $\{|c, n\rangle\}$ that diagonalizes $\hat{\mathbf{C}}^2$ and $\hat{C}_0$ simultaneously,

$$\mathbf{C}^2|c, n\rangle = c(c-1)|c, n\rangle, \qquad C_0|c, n\rangle = (c+n)|c, n\rangle, \tag{114}$$

where $c \in \mathbf{R}^+$ and $n \in \mathbf{N}_0$. From (113) and (114), we recognize that the modified radial harmonic oscillator under consideration is characterized by the constant,

$$c = \tfrac{1}{4}(2\gamma+3), \tag{115}$$

and that the spectrum of $\hat{H}_+$ is

$$E_n^{(+)} = 2(c+n) + g = 2n + 2\gamma + 2 + \varepsilon. \tag{116}$$

Since the Hamiltonian $\hat{H}_+$ is diagonalized on the basis that diagonalizes the operator $\hat{C}_0$, we identify the $su(1,1)$ states $|c, n\rangle$ characterized by (115) with the eigenstates $|\psi_n^+\rangle$ of $\hat{H}_+$. Thus, the ladder operators act on the SUSY states as

$$\hat{C}_+\big|\psi_n^{(+)}\big\rangle = \sqrt{(n+1)(n+\gamma+3/2)}\big|\psi_{n+1}^{(+)}\big\rangle, \tag{117}$$

$$\hat{C}_-\big|\psi_n^{(+)}\big\rangle = \sqrt{n(n+\gamma+1/2)}\big|\psi_{n-1}^{(+)}\big\rangle. \tag{118}$$

Next, we define the operators

$$\hat{D}_0 = \tfrac{1}{2}\hat{H}_- = \tfrac{1}{2}\hat{A}^\dagger\hat{A}, \qquad \hat{D}_\pm = \hat{A}^\dagger\hat{C}_\pm\hat{A}. \tag{119}$$

Use of (101), (117) and (118) enables us to show that $\hat{D}_\pm$, when acting on the SUSY states $|\psi_n^{(-)}\rangle$, behave like the ladder operators,

$$\hat{D}_+|\psi_n^{(-)}\rangle = \sqrt{E_n^{(-)}}\sqrt{(n+1)(n+\gamma+3/2)}\sqrt{E_{n+1}^{(+)}}|\psi_{n+1}^{(-)}\rangle \tag{120}$$

and

$$\hat{D}_-|\psi_n^{(-)}\rangle = \sqrt{E_n^{(-)}}\sqrt{n(n+\gamma+1/2)}\sqrt{E_{n-1}^{(+)}}|\psi_{n-1}^{(-)}\rangle. \tag{121}$$

What we wish to stress here is that the operators introduced by (119) form a cubic algebra,

$$[\hat{D}_0, \hat{D}_\pm] = \pm\hat{D}_\pm, \qquad [\hat{D}_+, \hat{D}_-] = -2(g^2 - (2c-1)^2 + 1)\hat{D}_0 + 12g\hat{D}_0^2 - 16\hat{D}_0^3, \tag{122}$$

where $g = \gamma + \varepsilon + 1/2$ and $c = (2\gamma+3)/4$. This algebra contains a quadratic term. It is not certain whether the representation we have constructed for the odd-polynomial algebra $su_{2p-1}(1,1)$ in section 2 is applicable to this case. Therefore, we select the parameter ε such that $g = 0$. Then we have the odd-polynomial cubic $su(1,1)$ algebra of interest,

$$[\hat{D}_0, \hat{D}_\pm] = \pm\hat{D}_\pm, \qquad [\hat{D}_+, \hat{D}_-] = -\big(\tfrac{3}{2} - 2\gamma(\gamma+1)\big)\hat{D}_0 - 16\hat{D}_0^3 \tag{123}$$

provided that

$$3 - 4\gamma(\gamma+1) > 0. \tag{124}$$

The two conditions (108) and (124) lead us to the restrictions on ε or γ,

$$-1 < \varepsilon < \tfrac{1}{2} \qquad \text{or} \qquad 0 < \gamma < \tfrac{1}{2}, \tag{125}$$

under which we shall work from now on.

By comparing (123) with the cubic algebra (88), we determine the parameters of (87)

$$\alpha_1 = \tfrac{3}{4} - \gamma(\gamma+1) \qquad \alpha_2 = 4, \tag{126}$$

from which follows the structure function

$$\Phi(x) = \left\{\tfrac{3}{4} - \gamma(\gamma+1)\right\} x + 4x^2. \tag{127}$$

From (16) the Casimir operator for the cubic algebra (123) is given by

$$\mathbf{D}^2 = -\hat{D}_+\hat{D}_- + \left\{\tfrac{3}{4} - \gamma(\gamma+1)\right\} \hat{D}_0(\hat{D}_0+1) + 4\hat{D}_0^2(\hat{D}_0+1)^2. \tag{128}$$

With the basis $\{|d, n\rangle\}$, we diagonalize $\hat{D}_0$ in (123) and the Casimir operator $\mathbf{D}^2$ of the cubic algebra as

$$\mathbf{D}^2|d, n\rangle = d(d-1)|d, n\rangle, \qquad \hat{D}_0|d, n\rangle = (d+n)|d, n\rangle, \tag{129}$$

where $d \in \mathbf{R}^+$ and $n \in \mathbf{N}_0$. Since the operator $\hat{H}_-$ is also diagonalized, we consider the $su_3(1, 1)$ states $|d, n\rangle$ as the eigenstates of $\hat{H}_-$ yielding the spectrum,

$$E_n^{(-)} = 2n + 2d. \tag{130}$$

In broken SUSY, as is mentioned above, the spectra of the partner Hamiltonians are identical, that is, $E_n^{(+)} = E_n^{(-)} = E_n$. Hence, comparing (116) and (130) with the condition $g = 0$, we have

$$E_n = 2n + \gamma + \tfrac{3}{2}, \qquad (n = 0, 1, 2, \ldots). \tag{131}$$

This implies that the representation space of $su_3(1, 1)$ is characterized by the constant

$$d = \tfrac{1}{4}(2\gamma + 3). \tag{132}$$

In this regard, we may identify the base states $|d, n\rangle$ of the cubic algebra (123) with the eigenstates $|\psi_n^{(-)}\rangle$ of $\hat{H}_-$. Even though the characteristic constant d of the representation of the cubic algebra (123) coincides with the characteristic constant c, given by (115), of the $su(1, 1)$ algebra (111), the two states $|c, n\rangle$ and $|d, n\rangle$ are distinct; namely, as we have identified in the above,

$$\left|\psi_n^{(+)}\right\rangle = |c, n\rangle, \qquad \left|\psi_n^{(-)}\right\rangle = |d, n\rangle \tag{133}$$

which are related by (101).

Substitution of the values (126) and $2k - 1 = \gamma + 1/2 (k = d)$ into (91) yields

$$a_\pm = -\tfrac{1}{2}\left(\gamma + \tfrac{1}{2}\right) \pm \tfrac{1}{2}. \tag{134}$$

The corresponding deformation factor is

$$\chi_n = \left(2n + \gamma - \tfrac{1}{2}\right)\left(2n + \gamma + \tfrac{3}{2}\right), \tag{135}$$

which turns out to be

$$\chi_n = E_{n-1}E_n, \tag{136}$$

where $E_n = E_n^{(+)} = E_n^{(-)}$. The structure factor is written as

$$\phi_n = n\left(n + \gamma + \tfrac{1}{2}\right) E_n E_{n-1}. \tag{137}$$

Therefore, with $d = (2\gamma + 3)/4$, we have

$$\hat{D}_+|d, n\rangle = \sqrt{(n+1)(n+\gamma+3/2)E_nE_{n+1}}|d, n+1\rangle \tag{138}$$

$$\hat{D}_-|d, n\rangle = \sqrt{n(n+\gamma+1/2)E_nE_{n-1}}|d, n-1\rangle, \tag{139}$$

which are consistent with the SUSY relations (120) and (121).

6.3. Coherent states for the conditionally solvable oscillator

Utilizing the deformation factor (135) we obtain

$$[\chi_n]! = 4^n \left(\tfrac{1}{2}\gamma + \tfrac{3}{4}\right)_n \left(\tfrac{1}{2}\gamma + \tfrac{7}{4}\right)_n \tag{140}$$

with which we can construct two sets of coherent states as follows.

Coherent states of the BG-type: Since the generalized factorial of the structure factor can be written as

$$[\phi_n]! = n!(\gamma + 3/2)_n[\chi_n]!, \tag{141}$$

substitution of (140) into (141) yields

$$[\phi_n]! = 2^{2n} n!(\gamma + 3/2)_n(\gamma/2 + 3/4)_n(\gamma/2 + 7/4)_n. \tag{142}$$

Inserting (142) into (62) we have the coherent states for the cubic algebra of the modified radial oscillator

$$|\xi\rangle = N_2^{-1}(|\xi|) \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!(\gamma + 3/2)_n(\gamma/2 + 3/4)_n(\gamma/2 + 7/4)_n}} \left(\frac{\xi}{2}\right)^n |\psi_n^{(-)}\rangle \tag{143}$$

with the normalization

$$N_2^2(|\xi|) = {}_0F_3\left(\gamma + 3/2, (2\gamma + 3)/4, (2\gamma + 7)/4; |\xi|^2/4\right). \tag{144}$$

It is apparent that the above coherent states are temporarily stable in Klauder's sense [17] that they evolve with the effective Hamiltonian,

$$\hat{\mathcal{H}} = \tfrac{1}{2}\omega\hbar\left(\hat{H}_- - \gamma - \tfrac{3}{2}\right) \tag{145}$$

as

$$\mathrm{e}^{-\mathrm{i}\hat{\mathcal{H}}t/\hbar}|\xi\rangle = |\xi\,\mathrm{e}^{-\mathrm{i}\omega t}\rangle. \tag{146}$$

The weight function for the resolution of unity (44) is

$$\begin{aligned}\rho(|\xi|^2) = {} & [4\Gamma(\gamma + 3/2)\Gamma(\gamma/2 + 3/4)\Gamma(\gamma/2 + 7/4)]^{-1} \\ & \times G_{04}^{40}\left(|\xi|^2/4\,|0, \gamma + 1/2, (2\gamma - 1)/4, (2\gamma + 3)/4\right).\end{aligned} \tag{147}$$

The coherent states obtained here are basically equivalent to those proposed by Junker and Roy [8] if $\varepsilon = -\gamma - 1/2$. Figure 1 shows the above weight function for the allowed range of parameter γ.

Coherent states of the P-type: With the same deformation factor (140), the coherent states of the P-type for the cubic case follow from (77),

$$|\eta\rangle = N_2^{-1}(|\eta|) \sum_{n=0}^{\infty} \left[\frac{(\gamma + 3/2)_n}{n!(\gamma/2 + 3/4)_n(\gamma/2 + 7/4)_n}\right]^{1/2} \left(\frac{\eta}{2}\right)^n |\psi_n^{(-)}\rangle, \tag{148}$$

with

$$N_2^2(|\eta|) = {}_1F_2\left(\gamma + 3/2; \gamma/2 + 3/4, \gamma/2 + 7/4; |\eta|^2/4\right). \tag{149}$$

These coherent states are also temporarily stable under the time evolution with the Hamiltonian $\hat{\mathcal{H}}$, that is,

$$\mathrm{e}^{-\mathrm{i}\hat{\mathcal{H}}t/\hbar}|\eta\rangle = |\eta\,\mathrm{e}^{-\mathrm{i}\omega t}\rangle. \tag{150}$$

The resolution of unity is achieved with the weight function,

$$\rho(|\eta|^2) = \frac{\Gamma(\gamma + 3/2)}{4\Gamma(\gamma/2 + 3/4)\Gamma(\gamma/2 + 7/4)} G_{13}^{30}\left(\frac{|\eta|^2}{4}\middle|\begin{matrix}\gamma + 1/2 \\ 0, (2\gamma - 1)/4, (2\gamma + 3)/4\end{matrix}\right), \tag{151}$$

which is shown in figure 2. The P-type coherent states are of course different from the BG-type states.

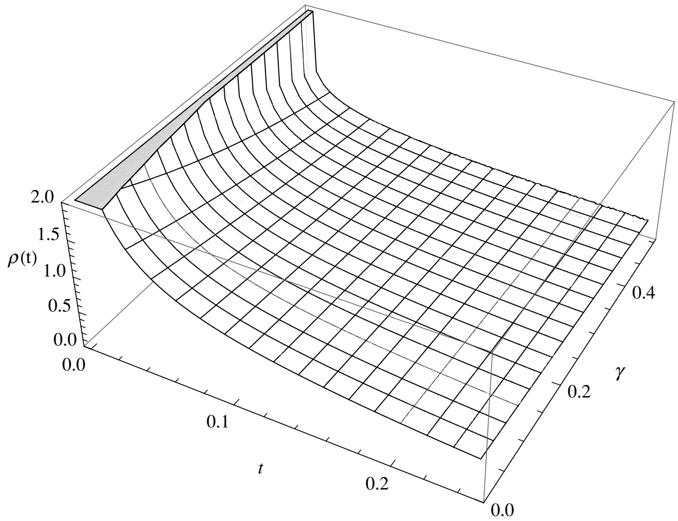

Figure 1. The weight function $\rho(t)$ of equation (147) for the Barut–Girardello coherent states with $t = |\xi|^2$, which is plotted for the allowed range of the characteristic parameter γ of the conditionally solvable oscillator.

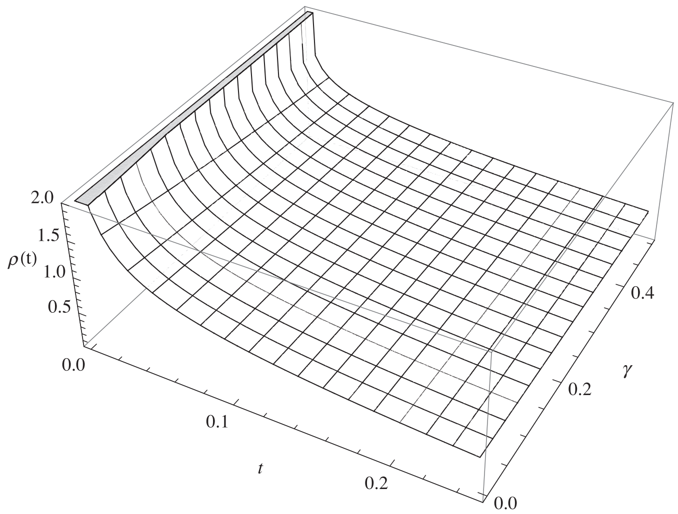

Figure 2. The weight function $\rho(t)$ of equation (151) for the Perelomov-type coherent states with $t = |\eta|^2$, plotted for the same oscillator.

7. Concluding remarks

Extending the deformed algebra $su_\Phi(2)$ of Bonatos, Danskaloyannis and Kolokotronis to $su_\Phi(1, 1)$ by a simple analytic continuation, and imposing the polynomial condition on the structure function, we have proposed a unified way to construct a discrete set of coherent states of the Barut–Girardello type and of the Perelomov type for the polynomial $su(1, 1)$ algebra.

We have also studied the connection between the cubic algebra $su(1, 1)$ and the conditionally solvable oscillator with broken SUSY. We found that the eigenstates of the Hamiltonian $\hat{H}_+$ in SUSY quantum mechanics can be identified with a standard basis of the $su(1, 1)$ algebra whereas the set of eigenstates of the other partner Hamiltonian $\hat{H}_-$ is identified with a representation space of the cubic algebra $su_3(1, 1)$. Then we construct coherent states of the Barut–Girardello type and of the Perelomov type for the conditionally solvable system.

The procedure used in this paper works only for polynomials of odd degree. In order to accommodate a polynomial algebra of even degree, such as the quadratic algebra, we have to modify the approach. Although our consideration is focused on the discrete class, a question remains open as to whether the same procedure may be extended to a continuous class in a way similar to that of an earlier work [19].

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